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Heat conduction in a symmetric body subjected to a current flow of symmetric input and output

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Abstract

Steady heat conduction in symmetrical electro-thermal problems is analyzed under the influence of a steady direct current passing through symmetrical regions of the boundary. In the present approach, solution is obtained by dividing the temperature field of the electro-thermal problem into two fields—one is related to the heat conduction problem without Joule heating and the other corresponds to a symmetric temperature field related to Joule heating induced by current supply. A Joule heating residue vector is introduced in the present analysis, which is expressed as a summation of the real heat flux and a vector representing the effect of Joule heating. It is shown that the Joule heating residue vector of symmetrical electro-thermal problem is related to the gradient of the temperature field associated with the problem without Joule heating. Moreover, the results of the present analysis show that, even if the Joule heating residue vector assumes a non-zero value, the temperature along the symmetric axis/plane remains constant when the applied current field is accompanied with antisymmetric heat fluxes on the boundary.

Keywords: Electro-thermal problem; Joule heating; Symmetric body; Heat conduction

1. Introduction

If an electrically conductive material is subjected to current flow, Joule heating occurs [1–3]. Nowadays, it is becoming an urgent need to deal with this type of electro-thermal problems and determine the associated resultant temperature fields properly, as far as the reliability of electronic devices is concerned. As a typical example of electro-thermal phenomena, one can cite electromigration [4–6], which is the phenomenon of atomic diffusion due to current flow. Abé et al. [7] have recently reviewed their works carried out on electromigration failure of metal lines; Saka and Ueda [8] have proposed a technique to form metallic nanowires utilizing the phenomenon of electromigration.

So far, one-dimensional problem of heat conduction in a wire under the influence of current flow has well been explained by Carslaw and Jaeger [2]. Regarding two-dimensional problems, Saka and Abé [9] have analyzed heat conduction in a cracked plate under a direct current field, and Sasagawa et al. [10] have extended the analysis to the problem of an angled metal line subjected to direct current flow. On the other hand, Greenwood and Williamson [1] have treated a conductor of general shape subjected to a direct current, and shown that equipotentials are isothermals under the assumptions that the relationship between the temperature and electrical potential at the positions of current input and output satisfies appropriate conditions (Joule heating residue vector, introduced in the present paper, is zero) and the remaining boundary is insulated both electrically and thermally. Greenwood and Williamson [1] have considered the case of temperature-dependent material properties for the above mentioned electro-thermal problem. It has however been shown that the above electro-thermal problem in which the thermal and electrical conductivities vary with temperature can be reduced to the corresponding problem with constant conductivities.

The present paper reports a simple theoretical approach for analyzing steady-state heat conduction in a uniform conductive solid subjected to steady direct current flow in terms of the associated temperature field. The development of the formulation is demonstrated here primarily with the help of a two-

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dimensional problem, however its extension to the case of a three-dimensional problem is also outlined. The shape of the two-dimensional body (for example, plate) is assumed to be symmetric to an axis where the locations of current input and output on the boundary are also symmetric to the same axis. The opposing surfaces of the plate are assumed to be insulated both electrically and thermally. In the present paper, by separating the temperature field into two components, attempt is made to interpret the associated temperature field of the symmetric electro-thermal problem in relation to the Joule heating residue vector which is expressed as a summation of heat flux due to temperature gradient and a vector representing the effect of Joule heating. And, as a result, the temperature distribution, especially along the symmetric axis is well identified with respect to a number of cases of practical interest. Finally, two different symmetric electro-thermal problems are solved by the standard analytical and numerical methods of solution, and the corresponding results are compared with those obtained by the present method of analysis. It is noted that, in our previous researches [9,10], an asymptotic solution has been investigated for the cases of cracked plate and angled line, however, the present solution is not limited to the cases mentioned, rather it involves a general treatment for solving symmetrical electro-thermal problems and is free from the limitation of asymptotic solution.

2. Fundamental equations of heat conduction for electro-thermal problem

When an electrically conductive material is subjected to a steady direct current flow, Joule heating occurs, which, in turn, causes increase in temperature in the material. By introducing a two-dimensional Cartesian coordinate system (x_1, x_2), the Poisson's equation for steady-state heat conduction in an electrically heated homogeneous isotropic conductive material is written as

$$\lambda\nabla^2(T - T_0) + \rho J_\alpha J_\alpha = 0, \tag{1}$$

where $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, J_α is the component of the current density vector $\mathbf{J}$, T the temperature, T_0 the temperature at a reference point in the material; λ and ρ are the thermal conductivity and electrical resistivity of the material, respectively. The Greek indices α take the values 1 and 2 and follow the summation convention.

The second term in the left-hand side of Eq. (1) represents the heat energy developed due to electrical heating, in unit volume and time. The material properties λ and ρ are assumed to be constants throughout the analysis.

The Ohm's law in terms of electrical potential is written as

$$\rho\mathbf{J} = -\operatorname{grad}(\phi - \phi_0), \tag{2}$$

where ϕ is the electrical potential, and ϕ_0 is the electrical potential at a reference point in the material.

With the help of Eq. (2) and the law of current conservation, Eq. (1) can now be transformed to the Laplace equation [1,9] as follows:

$$\nabla^2\left\{(T - T_0) + (\phi - \phi_0)^2/(2\lambda\rho)\right\} = 0. \tag{3}$$

Table 1
Analogy between electric and electro-thermal problems

Electric problem	Electro-thermal problem
$\phi - \phi_0$	$(T - T_0) + (\phi - \phi_0)^2/(2\lambda\rho)$
$\nabla^2(\phi - \phi_0) = 0$	$\nabla^2\{(T - T_0) + (\phi - \phi_0)^2/(2\lambda\rho)\} = 0$
$J_\alpha = -\frac{1}{\rho}\frac{\partial(\phi-\phi_0)}{\partial x_\alpha}$	$p_\alpha = -\lambda\frac{\partial\{(T-T_0)+(\phi-\phi_0)^2/(2\lambda\rho)\}}{\partial x_\alpha}$
$\mathbf{J}$	$\mathbf{p}$
ρ	$1/\lambda$

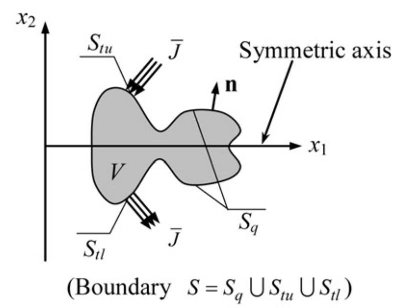

Fig. 1. Geometry of a symmetric plate subjected to symmetric current input and output in a two-dimensional Cartesian coordinate system (x_1, x_2).

The Fourier's law of heat conduction is given by

$$\mathbf{q} = -\lambda\operatorname{grad}T, \tag{4}$$

where $\mathbf{q}$ is the heat flux due to temperature gradient, and it is the real heat-flux vector.

Based on Eq. (3), let us define the vector related to electro-thermal problem, $\mathbf{p}$ [9], as follows:

$$\mathbf{p} = -\lambda\operatorname{grad}\left\{(T - T_0) + (\phi - \phi_0)^2/(2\lambda\rho)\right\}. \tag{5}$$

The first term in the right-hand side of Eq. (5), i.e., $-\lambda\operatorname{grad}(T - T_0)$, represents the real heat flux and the second term, $-\lambda\operatorname{grad}\{(\phi - \phi_0)^2/(2\lambda\rho)\}$, reflects the corresponding effect of Joule heating. The vector $\mathbf{p}$ is thus realized to be a fictitious heat flux.

Here, the parameters $(\phi - \phi_0)$ and $(T - T_0) + (\phi - \phi_0)^2/(2\lambda\rho)$, $\mathbf{J}$ and $\mathbf{p}$, ρ and $1/\lambda$ are the mathematical equivalents for different problems, respectively. Table 1 shows the analogy of these parameters between the electric and electro-thermal problems.

Let us consider a symmetric domain of interest, V, which is enclosed by the boundary S, as shown in Fig. 1. Referring to Fig. 1, the associated boundary conditions of the electro-thermal problem can be expressed as follows:

(a) Thermal boundary conditions:

$$-\lambda\frac{\partial T}{\partial n} = \bar{q}(x_1, x_2) \quad \text{on } S_q, \tag{6a}$$

$$T = \bar{T}_u(x_1, x_2) \quad \text{on } S_{tu} \tag{6b}$$

and

$$T = \bar{T}_l(x_1, x_2) \quad \text{on } S_{tl}; \tag{6c}$$

(b) Electrical boundary conditions:

$$J_n = 0 \quad \text{on } S_q, \tag{7a}$$

$$J_n = -\bar{J}(x_1, x_2) \quad \text{on } S_{tu} \tag{7b}$$

and

$$J_n = \bar{J}(x_1, x_2) \quad \text{on } S_{tl}, \tag{7c}$$

where $\bar{q}(x_1, x_2)$ is a given distribution of heat flux, $\partial/\partial n$ the directional derivative along the unit outward normal $\mathbf{n}$, $\bar{T}_u(x_1, x_2)$ and $\bar{T}_l(x_1, x_2)$ the given distributions of T on S_{tu} and S_{tl}, respectively, J_n the normal component of the current density vector $\mathbf{J}$, and $\bar{J}(x_1, x_2)$ the given distribution of J_n on the boundary. The boundaries S_{tu} and S_{tl} correspond to the segments of S $(= S_q \cup S_{tu} \cup S_{tl})$ from which input and output current pass, and S_q is electrically insulated. Note that the boundary conditions given by Eqs. (6) and (7) are not limited to a special case, rather cover almost all the cases of electro-thermal problems in practice.

3. Analysis of symmetric problems

This section analyzes the temperature field of the symmetric two-dimensional problem. The current density $\mathbf{J}$ is applied to a plate which is symmetric about the axis x_1. The corresponding positions of current input and output on the boundary are also symmetric to axis x_1. The distribution of the input current density is the same as that of the output one $[\bar{J}(x_1, x_2) = \bar{J}(x_1, -x_2)]$. Fig. 1 shows the flow of $\mathbf{J}$ through two symmetrical regions of the plate boundary. The temperature field T is assumed to be composed of two fields—one is related to the heat conduction problem without Joule heating and the other corresponds to a symmetric temperature field related to Joule heating caused by current supply. In addition, T_0 and ϕ_0 are assumed to be the temperature and electrical potential at a reference point on the symmetric axis, respectively. The present paper deduces appropriate results of the temperature field owing to the division of the temperature field T.

3.1. Division of the temperature field

The temperature field T is divided into two components as follows:

$$T = T^* + T^{**}, \tag{8}$$

where the temperature field T^* is due to input and output of heat fluxes on the boundary of the plate, and T^{**} reflects the effect of Joule heating under the condition of thermal insulation for the segment S_q, and for this temperature field T^{**}, the segments of current input and output (i.e., S_{tu} and S_{tl}) are not thermally insulated, rather they are identified in terms of known temperatures. The temperature field T^{**} is symmetric to the axis x_1, but not the symmetric component of T. On the other hand, the temperature field T^* can be symmetric or asymmetric. By considering an example case of a plate subjected to the heat flux given parallelly to the axis x_1, it can easily be shown that T^* can be symmetric to axis x_1.

From Eqs. (3) and (8) it can be shown that the temperature fields T^* and T^{**} are the solutions of the following two differential equations, respectively:

$$\nabla^2\big(T^* - T_0^*\big) = 0 \tag{9}$$

and

$$\nabla^2\big\{(T^{**} - T_0^{**}) + (\phi - \phi_0)^2/(2\lambda\rho)\big\} = 0, \tag{10}$$

where $(\phi - \phi_0)^2/(2\lambda\rho)$ is symmetric to the x_1-axis. Quantities T_0^* and T_0^{**} are the values of T^* and T^{**} at the reference point on the symmetric axis, respectively.

The necessary boundary conditions for solving the temperature field T^* from Eq. (9) are as follows:

$$-\lambda\frac{\partial T^*}{\partial n} = \bar{q}(x_1, x_2) \quad \text{on } S_q, \tag{11a}$$

$$T^* = \bar{T}_u^*(x_1, x_2) \quad \text{on } S_{tu} \tag{11b}$$

and

$$T^* = \bar{T}_l^*(x_1, x_2) \quad \text{on } S_{tl}, \tag{11c}$$

where $\bar{T}_u^*(x_1, x_2)$ and $\bar{T}_l^*(x_1, x_2)$ are given distributions of T^* on S_{tu} and S_{tl}, respectively.

The corresponding conditions for solving the symmetric temperature field T^{**} from Eq. (10) are as follows:

$$\frac{\partial}{\partial n}\big[(T^{**} - T_0^{**}) + (\phi - \phi_0)^2/(2\lambda\rho)\big] = 0 \quad \text{on } S_q \tag{12a}$$

and

$$\begin{aligned}&\big(T^{**} - T_0^{**}\big) + (\phi - \phi_0)^2/(2\lambda\rho)\\ &\quad = \begin{cases}(\bar{T}^{**} - T_0^{**}) + (\bar{\phi}_u - \phi_0)^2/(2\lambda\rho) & \text{on } S_{tu},\\ (\bar{T}^{**} - T_0^{**}) + (\bar{\phi}_l - \phi_0)^2/(2\lambda\rho) & \text{on } S_{tl},\end{cases}\end{aligned} \tag{12b}$$

where $\bar{T}^{**}(x_1, x_2)$ is the given distribution of T^{**} and is applied to S_{tu} and S_{tl} symmetrically to the axis $x_1[\bar{T}^{**}(x_1, x_2) = \bar{T}^{**}(x_1, -x_2)]$. And $\bar{\phi}_u(x_1, x_2)$ and $\bar{\phi}_l(x_1, x_2)$ are the distributions of electrical potential at S_{tu} and S_{tl}, respectively, of which combination $\{\bar{\phi}_u(x_1, x_2) - \bar{\phi}_l(x_1, -x_2)\}$ $[= 2\{\bar{\phi}_u(x_1, x_2) - \phi_0\} = 2\{\phi_0 - \bar{\phi}_l(x_1, -x_2)\}]$ corresponds to the given distribution of current density $\bar{J}(x_1, x_2)$.

For the case of our present symmetric problem, the boundary conditions of T^{**} field given by Eqs. (12a) and (12b) reduce to

$$\frac{\partial T^{**}}{\partial n} = 0 \quad \text{on } S_q \tag{13a}$$

and

$$T^{**} = \bar{T}^{**}(x_1, x_2) \quad \text{on } S_{tu} \text{ and } S_{tl}. \tag{13b}$$

Now, we have,

$$\bar{T}_u^*(x_1, x_2) + \bar{T}^{**}(x_1, x_2) = \bar{T}_u(x_1, x_2), \tag{14a}$$

$$\bar{T}_l^*(x_1, x_2) + \bar{T}^{**}(x_1, x_2) = \bar{T}_l(x_1, x_2) \tag{14b}$$

and

$$T_0^* + T_0^{**} = T_0. \tag{14c}$$

Now, likewise the case of Eqs. (9) and (10), Eq. (5) can be divided as

$$\mathbf{p}^* = -\lambda\,\mathrm{grad}\big(T^* - T_0^*\big) \tag{15}$$

and

$$\mathbf{p}^{**} = -\lambda \operatorname{grad}\{(T^{**} - T_0^{**}) + (\phi - \phi_0)^2/(2\lambda\rho)\}, \tag{16}$$

where $\mathbf{p}^*$ and $\mathbf{p}^{**}$ are concerned with the temperature fields T^* and T^{**}, respectively.

As far as the temperature field T^{**} is concerned, it is known that the boundary segment S_q is insulated electrically and thermally, and thus we can get $\mathbf{p}^{**} = 0$ on S_q. That is, a possible non-zero $\mathbf{p}^{**}$ may exist only on S_{tu} and S_{tl} segments of the boundary. In the following, it will be demonstrated that $\mathbf{p}^{**} = 0$ is also valid on S_{tu} and S_{tl}, and within the body.

First, let us consider the simple case that the current density $\mathbf{J}$ only passes through two symmetric points on the boundary. For the symmetric problem, it is known that both $T^{**} - T_0^{**}$ and $(\phi - \phi_0)^2/(2\lambda\rho)$ are symmetric to the x_1-axis. Thus we can obtain that $\mathbf{p}^{**}$ is also symmetric to the x_1-axis from Eq. (16). That is to say, the vectors $\mathbf{p}^{**}$ either enter ($\mathbf{p}^{**} \cdot \mathbf{n} \leqslant 0$) or get out of the region ($\mathbf{p}^{**} \cdot \mathbf{n} \geqslant 0$) through the two symmetric points on the boundary simultaneously.

On the other hand, from Eqs. (10) and (16), we get

$$\operatorname{div} \mathbf{p}^{**} = 0. \tag{17}$$

And Gauss's theorem is given by

$$\int_V \operatorname{div} \mathbf{p}^{**} \, dV = \int_S \mathbf{p}^{**} \cdot \mathbf{n} \, dS = 0. \tag{18}$$

Thus we can obtain

$$\mathbf{p}^{**} = 0. \tag{19}$$

The symmetric input and output of the current density can readily be extended to the more general case that the current density $\mathbf{J}$ passes through several symmetric sub-segments on the boundary, with the help of superposition. The problem can be expressed as summation of several sub-problems in which $\mathbf{J}$ passes through a pair of symmetrical sub-segments of S_{tu} and S_{tl}. Eq. (19) is valid for each sub-problem. As a result, Eq. (19) is also valid for the general problem based on the principle of superposition.

Now, from Eqs. (5), (15), (16) and (19), we get $\mathbf{p} = \mathbf{p}^*$, which reveals that the vector $\mathbf{p}$ of a symmetric electro-thermal problem is basically related to the gradient of the temperature field T^*, and is free from the effect of Joule heating. That is to say, the vector $\mathbf{p}$ is a part of the real heat flux. So we can name it Joule heating residue vector.

Next, substitution of Eq. (16) into Eq. (19) gives

$$(T^{**} - T_0^{**}) + (\phi - \phi_0)^2/(2\lambda\rho) = C_1, \tag{20}$$

where C_1 is a constant. Note that, at the reference point on the symmetric axis, $T^{**} = T_0^{**}$ and $\phi = \phi_0$, so we get

$$C_1 = 0. \tag{21}$$

Substitution of Eq. (21) into Eq. (20) gives

$$T^{**} = T_0^{**} - (\phi - \phi_0)^2/(2\lambda\rho). \tag{22}$$

It is noted here that the symmetric temperature field given by Eq. (22) takes a form similar to that obtained by Greenwood and Williamson [1] and thus the problem associated with the temperature field T^{**} resembles the Greenwood and Williamson's one.

Further, from Eq. (22) we can get

$$\frac{\partial T^{**}}{\partial n} = -\frac{(\phi - \phi_0)}{\lambda\rho}\frac{\partial(\phi - \phi_0)}{\partial n}. \tag{23}$$

Since we have $\partial T^{**}/\partial n = 0$ on S_q [Eq. (13a)], Eq. (23) gives $\partial(\phi - \phi_0)/\partial n = 0$ on S_q. Therefore, the thermal and electrical boundary conditions of the electro-thermal problem governed by Eq. (10) are not independent, rather they have to maintain certain functional relation on the boundary as suggested by Eq. (23).

3.2. Temperature field T^{**} on the symmetric axis

There is no current flow along the symmetric axis x_1, so the electrical potential along the axis is constant, that is, $\phi = \phi_0$. According to Eq. (22), it is thus known that

$$T^{**} = T_0^{**}, \tag{24}$$

which is valid for any point on the symmetric axis.

3.3. Temperature fields T^* and T in the case of $\mathbf{p} = 0$

For the symmetric region, consider the situation of $\mathbf{p} = 0$. According to Eq. (15) and $\mathbf{p} = \mathbf{p}^*$, for the above case, we get

$$T^* - T_0^* = C_2, \tag{25}$$

where C_2 is a constant. It is known that $T^* = T_0^*$ at the reference point on the symmetric axis, which implies that

$$C_2 = 0. \tag{26}$$

Therefore, $T^* = T_0^*$ for the entire region of the symmetric body.

From Eqs. (8), (22), (25) and (26), the expression for the temperature field T for any points within the body in the case of $\mathbf{p} = 0$ can be obtained as

$$T = T_0 - (\phi - \phi_0)^2/(2\lambda\rho). \tag{27}$$

From Eq. (27), equipotentials are isothermals as revealed by Greenwood and Williamson [1], and T_0 is the maximum temperature. The corresponding temperature field T on the symmetric axis ($\phi = \phi_0$) is obtained as $T = T_0$, which shows that the temperature on the symmetric axis is always constant in the case of $\mathbf{p} = 0$. This particular case of $\mathbf{p} = 0$ ($\mathbf{p}^* = \mathbf{p}^{**} = 0$), in fact, corresponds to the case when the given heat flux on the boundary S_q vanishes, that is, $\bar{q}(x_1, x_2) = 0$, and $\bar{T}_u(x_1, x_2) = \bar{T}_l(x_1, -x_2)$ on S_{tu} and S_{tl}; let us call the situation Case A.

3.4. Temperature fields T^* and T in the case of $\mathbf{p} \neq 0$

In the case of $\mathbf{p} \neq 0$, Eq. (27) cannot generally be valid. Indeed, if $\mathbf{p} \neq 0$ and hence $\mathbf{p}^* \neq 0$, we get the following solution from Eq. (15):

$$T^* - T_0^* = f(x_1, x_2), \tag{28}$$

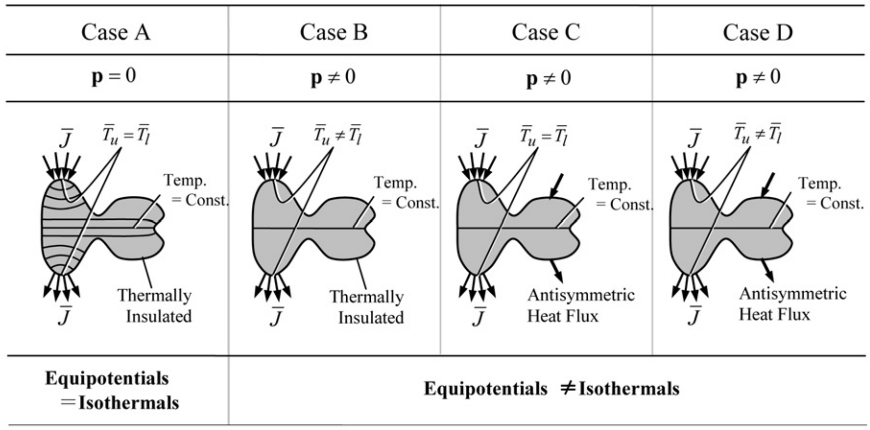

Fig. 2. Schematic illustration of the solution of four different cases of symmetric conductive plates having different thermal boundary conditions.

where $f(x_1, x_2)$ is a function of x_1 and x_2. Then from Eqs. (8), (22) and (28), we get the following equation for any points within the body:

$$T = T_0 - (\phi - \phi_0)^2/(2\lambda\rho) + f(x_1, x_2). \quad (29)$$

Eq. (29) differs from Eq. (27), and T is, in general, not constant on the symmetric axis.

Let us now consider a case of $\mathbf{p} \neq 0$, where, S_q is thermally insulated, that is, $\bar{q}(x_1, x_2) = 0$ on S_q, but temperatures on S_{tu} and S_{tl} are different $[\bar{T}_u(x_1, x_2) \neq \bar{T}_l(x_1, -x_2)]$. This situation is named as Case B. For this case, since $\bar{T}_u^*(x_1, x_2) \neq \bar{T}_l^*(x_1, -x_2)$, the following relation holds:

$$T^*(x_1, x_2) - T_0^* = T_0^* - T^*(x_1, -x_2). \quad (30)$$

Hence from Eq. (28), we get

$$f(x_1, x_2) = -f(x_1, -x_2). \quad (31)$$

Thus on the symmetric axis ($x_2 = 0$), we have

$$f(x_1, 0) = 0. \quad (32)$$

Eqs. (29) and (32) imply that temperature on the symmetric axis ($x_2 = 0$ and $\phi = \phi_0$) is constant ($T = T_0$), even in the case of $\mathbf{p} \neq 0$ and $\bar{q}(x_1, x_2) = 0$ on S_q. This situation holds also in the case in which the heat flux is input and output at symmetrical regions on S_q so that $\bar{q}(x_1, x_2)$ is antisymmetric with respect to axis x_1. For this case of $\bar{q}(x_1, x_2) \neq 0$, we may again consider two different cases of interest, as follows: (1) Case C: $\bar{q}(x_1, x_2) = -\bar{q}(x_1, -x_2) \neq 0$ on S_q and $\bar{T}_u(x_1, x_2) = \bar{T}_l(x_1, -x_2)$ on S_{tu} and S_{tl}, respectively; (2) Case D: $\bar{q}(x_1, x_2) = -\bar{q}(x_1, -x_2) \neq 0$ on S_q and $\bar{T}_u(x_1, x_2) \neq \bar{T}_l(x_1, -x_2)$ on S_{tu} and S_{tl}, respectively. It will be worthwhile to mention that, for the above cases just explained, equipotentials are not isothermals, because $f(x_1, x_2) \neq 0$ for $x_2 \neq 0$. It is noted that the condition of constant potential ($\phi = \phi_0$) along the symmetric axis can also be deduced following the similar procedure used to derive Eqs. (28) to (32). The results of the four cases (Cases A to D) considered in the present analysis along with their corresponding physical conditions on the boundary are illustrated schematically in a comparative fashion in Fig. 2; one can easily compare the solutions with respect to the boundary conditions as well as the Joule heating residue vector $\mathbf{p}$.

3.5. Solutions of example problems and comparison

This subsection demonstrates the usefulness of the present method of analysis through the solutions of a simple analytical example and a numerical example.

3.5.1. Analytical example

First, let us consider a simple electro-thermal problem in a thin lathy plate, as shown in Fig. 3(a). The plate has a rectangular shape with the length l and width w. And the thickness of the plate is t. Current is input and output uniformly from the upper and lower boundaries (that is, $x_2 = l/2$ and $x_2 = -l/2$), respectively, of which density is $\bar{J}$. And the temperatures of the upper and lower boundaries are $\bar{T}_u$ and $\bar{T}_l$, respectively. In addition, the left and right boundaries are insulated thermally and electrically. This problem falls under the category of Case B.

Noting that the temperature gradient in a lathy plate along the x_2-direction is much larger than that along the x_1-direction, we can replace $\nabla^2 T$ with $\partial^2 T/\partial x_2^2$. Thus the governing equation for the problem of Fig. 3(a) is

$$\lambda \frac{\partial^2 T}{\partial x_2^2} + \rho \bar{J}^2 = 0. \quad (33)$$

The boundary conditions for temperature are

$$T|_{x_2 = l/2} = \bar{T}_u \quad (34)$$

and

$$T|_{x_2 = -l/2} = \bar{T}_l. \quad (35)$$

The solution of Eq. (33) with regard to the boundary conditions (34) and (35) are obtained as follows:

$$T = -\frac{\rho}{2\lambda}\bar{J}^2 x_2^2 + \frac{1}{l}(\bar{T}_u - \bar{T}_l)x_2 + \frac{1}{2}(\bar{T}_u + \bar{T}_l) + \frac{\rho l^2}{8\lambda}\bar{J}^2. \quad (36)$$

Next, according to the present method of analysis, the problem of Fig. 3(a) is divided into two sub-problems as shown in Fig. 3 (b) and (c).

For the problem of Fig. 3(b), no current is applied to the plate, and the temperatures of the upper and lower boundaries

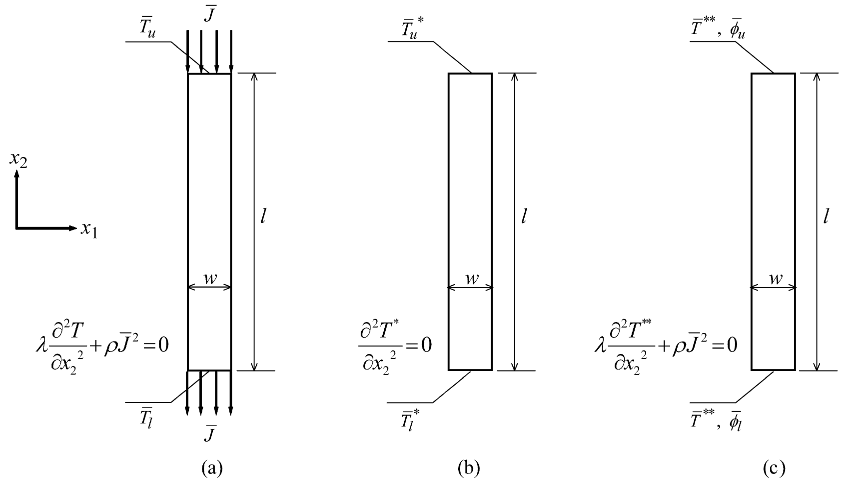

Fig. 3. Illustration of the analytical example of a rectangular plate: Problem (a) = Problem (b) + Problem (c).

are $\bar{T}_u^*$ and $\bar{T}_l^*$, respectively. Now, the governing equation for the problem of Fig. 3(b) is

$$\frac{\partial^2 T^*}{\partial x_2^2} = 0. \tag{37}$$

And the boundary conditions are

$$T^*|_{x_2=l/2} = \bar{T}_u^* \tag{38}$$

and

$$T^*|_{x_2=-l/2} = \bar{T}_l^*. \tag{39}$$

We can get the solution of Eq. (37) as

$$T^* = \frac{1}{l}\left(\bar{T}_u^* - \bar{T}_l^*\right)x_2 + \frac{1}{2}\left(\bar{T}_u^* + \bar{T}_l^*\right). \tag{40}$$

For the problem of Fig. 3(c), the upper and lower boundaries have the same temperature $\bar{T}^{**}$ and their electrical potentials are $\bar{\phi}_u$ and $\bar{\phi}_l$, respectively. Now, the governing equation for the problem of Fig. 3(c) is

$$\lambda \frac{\partial^2 T^{**}}{\partial x_2^2} + \rho \bar{J}^2 = 0. \tag{41}$$

And the boundary conditions are

$$T^{**}|_{x_2=l/2} = \bar{T}^{**}, \tag{42}$$

$$T^{**}|_{x_2=-l/2} = \bar{T}^{**}, \tag{43}$$

$$\phi|_{x_2=l/2} = \bar{\phi}_u \tag{44}$$

and

$$\phi|_{x_2=-l/2} = \bar{\phi}_l. \tag{45}$$

The solution of this problem is shown in Eq. (22).

According to the Ohm's law, we can get the following expressions about ϕ, ϕ_0, $\bar{\phi}_u$ and $\bar{\phi}_l$:

$$\bar{\phi}_u - \bar{\phi}_l = \bar{J}\rho l \tag{46}$$

and

$$\phi - \phi_0 = \bar{J}\rho x_2. \tag{47}$$

Substituting Eqs. (42), (44) and (46) into Eq. (22), we get

$$T_0^{**} = \bar{T}^{**} + \frac{\rho l^2}{8\lambda}\bar{J}^2. \tag{48}$$

Substitution of Eqs. (47) and (48) into Eq. (22) gives

$$T^{**} = \bar{T}^{**} + \frac{\rho \bar{J}^2}{8\lambda}\left(l^2 - 4x_2^2\right). \tag{49}$$

From Eqs. (36), (40) and (49), we can get

$$T = T^* + T^{**}, \tag{50}$$

which verifies the appropriateness of the present approach. In addition, the component of $\mathbf{p}$ in the x_2-direction, p_2, is obtained as follows:

$$p_2 = -\lambda \frac{\partial}{\partial x_2}\left\{(T - T_0) + (\phi - \phi_0)^2/(2\lambda\rho)\right\} = -\frac{\lambda}{l}(\bar{T}_u - \bar{T}_l), \tag{51}$$

which is identical to the component of $\mathbf{p}^*$ in the x_2-direction.

3.5.2. Numerical example

Consider a rectangular copper plate with a right-angled diamond-shaped hole at the center as shown in Fig. 4. The material properties of the plate are assumed to be $\rho = 17.54\ \mathrm{m\Omega \cdot \mu m}$ and $\lambda = 384\ \mathrm{W/(m\,K)}$. The length and width of the plate are 0.2 and 0.1 m, respectively, and its thickness is 1 mm. The diagonal length of the hole is 0.02 m. Current is input and output uniformly from the upper and lower boundaries, respectively, of which density $\bar{J}$ is $30\ \mathrm{MA/m^2}$. The left and right boundaries of the plate as well as the boundaries of the hole are assumed to be electrically insulated. The temperatures of the upper and lower boundaries are 328 and 273 K, respectively. The left boundary of the plate and the boundaries of the hole are thermally insulated. In addition, a heat flux $\bar{q} = 100\ \mathrm{kW/m^2}$ is input and output from the right boundary of the plate within the regions of $x_2 \in [0.05, 0.1]$ and $x_2 \in [-0.1, -0.05]$, respectively; the remaining part of the right boundary is thermally insulated. This problem falls under the category of Case D.

This problem is solved by finite element method using the COMSOL Multiphysics software package [11]. Fig. 5 shows

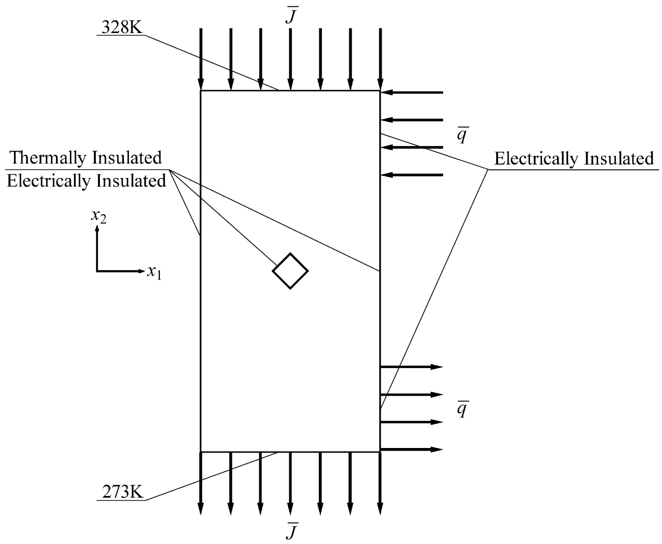

Fig. 4. Geometry and boundary conditions of a rectangular copper plate having a right-angled diamond-shaped hole at its center. The length and width of the plate are 0.2 and 0.1 m, respectively, and its thickness is 1 mm. The diagonal length of the hole is 0.02 m.

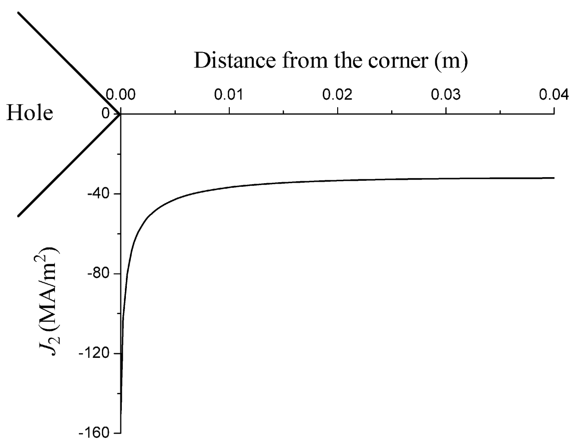

Fig. 5. Distribution of J_2 along the symmetric axis right to the corner of the rectangular copper plate.

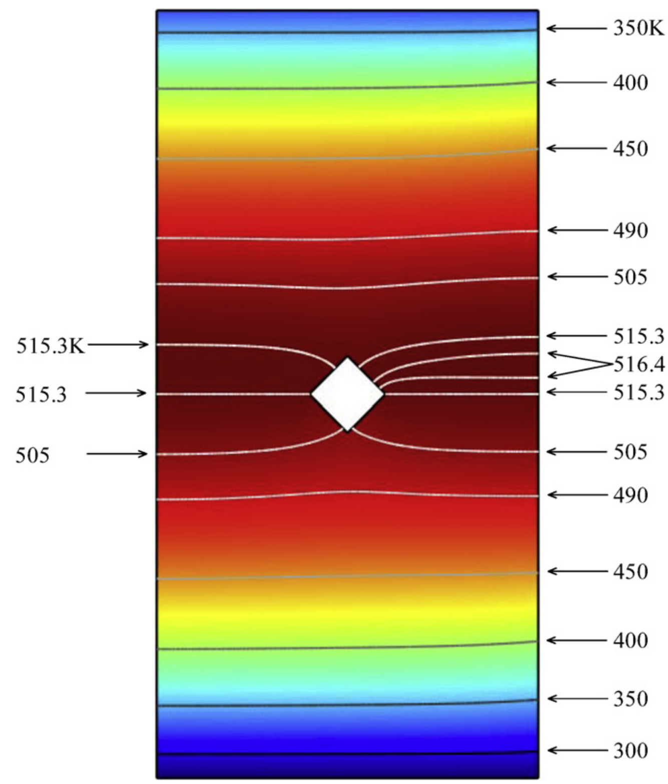

Fig. 6. Temperature distribution in the copper plate of Fig. 4 and the associated isotherms.

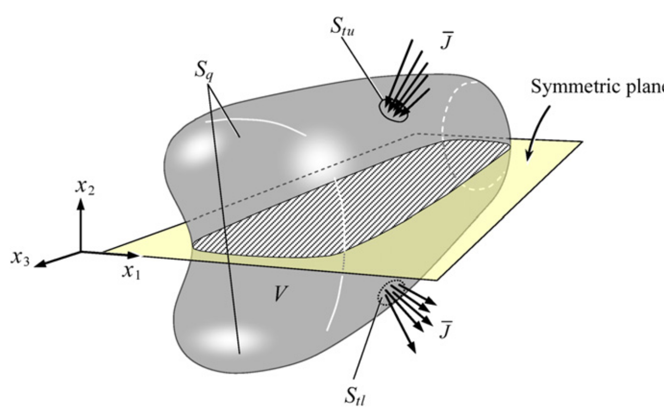

Fig. 7. Geometry of a symmetric three-dimensional body subjected to symmetric current input and output in Cartesian coordinate system (x_1, x_2, x_3).

the distribution of J_2, which is the x_2 component of current density, along the symmetric axis right to the corner. The abscissa shows the distance from the corner. The curve shows that J_2 concentrates near the corner. Fig. 6 shows the temperature distribution in the plate, and the lines specify the isotherms. It is shown that the temperature along the x_1-axis takes a constant value of 515.3 K.

3.6. General comments

It is known that if there is a sharp corner, a concentration of current density and thus Joule heating are induced [9,10]. However, as long as the plate is symmetric and the sharp corner exists on the symmetric axis, the temperature will still remain constant along the symmetric axis, provided that the problem satisfies the condition of $\mathbf{p} = 0$ (Case A) or Cases B, C and D when $\mathbf{p} \neq 0$. This has been verified by the solution of the numerical example discussed above.

Although the present treatment is demonstrated basically for the case of a symmetric plate subjected to a current of symmetric input and output, the treatment can easily be extended to the case of a three-dimensional body which is symmetric to a plane and the current input and output are also symmetric to the same plane. The case of such a three-dimensional symmetric body is schematically illustrated in Fig. 7.

Since the three-dimensional problem is also governed by the Laplace equation, following the similar procedure it can be shown that $\mathbf{p}^{**} = 0$ in V of the three-dimensional body and hence $\mathbf{p} = \mathbf{p}^*$. Hence T^{**} will be constant on the symmetric plane and, for the four cases mentioned above (Cases A to D), T will be constant on the symmetric plane.

4. Discussions and conclusions

Saka and Abé [9] considered a one-dimensional problem of heat conduction for a uniform straight bar subjected to direct current flow. It was shown that the vector **p** takes identical values at both ends of the bar. Sasagawa et al. [10], on the other hand, considered a two-dimensional problem of steady current flow near the corner of a homogeneous, isotropic, angled infinite metal line. The fictitious heat flux considering Joule heating at any point R far from the corner of the metal line was shown to be due to the temperature difference between the point R and its symmetric point with respect to the bisector of the angle.

The present symmetric analysis of the electro-thermal problem enables us to explain the meaning of the vector **p** in a simpler and general way. The steady-state heat conduction and the associated temperature field in a uniform symmetrical plate of general shape subjected to steady direct current passing through symmetrical regions of the boundary, were analyzed in a fashion so that the formulation can be extended to the case of symmetric three-dimensional electro-thermal problems. Dividing the temperature field of the electro-thermal problem into two components, T^* and T^{**}, it was possible to interpret the temperature field of the electro-thermal problem in terms of the Joule heating residue vector which gives the combined effect of temperature gradient and Joule heating. Moreover, the two problems associated with the above temperature fields were well characterized in terms of their respective governing differential equations as well as the physical conditions on the boundary. The vector **p** was shown to be related to the gradient of the temperature field T^*. Moreover, the results of the present analysis identified a number of cases of practical interest in terms of given heat flux and temperatures on the boundary, for which the temperature along the symmetric axis remains constant.

Saka and Abé [9] derived the temperature distribution near the crack tip by using the path-independent j_e and j_t integrals, which relate the distribution of electrical potential to the temperature field, and Sasagawa et al. [10] analyzed the temperature field by neglecting small terms with regard to the region near the corner in the derivation. As a result, both of their solutions were asymptotic. As compared to the previous researches, the present solution led to the temperature field which is free from the limitation of asymptotic solution. Moreover, the present analysis is not limited to the problems with a crack or a sharp corner, but is valid for all symmetrical problems. Further, in contrast with the problem treated by Greenwood and Williamson [1], the superiority of the present study lies on the point that the method of analysis is not only limited to the condition of thermal insulation on the boundary segment S_q, rather it can deal with given heat fluxes as well.

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References

[1] J.A. Greenwood, J.B.P. Williamson, Electrical conduction in solids. II. Theory of temperature-dependent conductors, Proc. Roy. Soc. Lond. A 246 (1244) (1958) 13–31.
[2] H.S. Carslaw, J.C. Jaeger, in: Conduction of Heat in Solids, second ed., Clarendon, Oxford, 1959, pp. 149–151.
[3] T. Sameshima, K. Ozaki, Crystallization of silicon thin films by current-induced Joule heating, Thin Solid Films 383 (1–2) (2001) 107–109.
[4] H.B. Huntington, A.R. Grone, Current-induced marker motion in gold wires, J. Phys. Chem. Solids 20 (1–2) (1961) 76–87.
[5] I.A. Blech, Electromigration in thin aluminum films on titanium nitride, J. Appl. Phys. 47 (4) (1976) 1203–1208.
[6] W. Wu, J.S. Yuan, S.H. Kang, A.S. Oates, Electromigration subjected to Joule heating under pulsed DC stress, Solid-State Electron. 45 (12) (2001) 2051–2056.
[7] H. Abé, K. Sasagawa, M. Saka, Electromigration failure of metal lines, Int. J. Fract. 138 (1–4) (2006) 219–240.
[8] M. Saka, R. Ueda, Formation of metallic nanowires by utilizing electromigration, J. Mater. Res. 20 (10) (2005) 2712–2718.
[9] M. Saka, H. Abé, Path-independent integrals for heat conduction analysis in electrothermal crack problems, J. Thermal Stresses 15 (1) (1992) 71–83.
[10] K. Sasagawa, M. Saka, H. Abé, Current density and temperature distributions near the corner of angled metal line, Mechanics Research Comm. 22 (5) (1995) 473–483.
[11] A.B. COMSOL, COMSOL Multiphysics User's Guide, Version 3.2, Stockholm, Sweden, 2005.